GeoELSEvp: a spectral element approach for dynamic elasto-viscoplastic problems

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Summary
A spectral element method for the numerical analysis of 2D and 3D wave propagation problems is equipped with a viscoplastic constitutive model in order to provide a tool able to grant the flexibility of a finite element technique in dealing with non linear constitutive models, together with the accuracy of spectral methods for propagating waves. The proposed tool is both fast and behaves like a black-box requiring minimum interaction to the user. The implementation of the method is illustrated and applied to a set of 2D reference cases.

1. Introduction
Wave propagation phenomena can be easily studied nowadays thanks to powerful numerical techniques and probably the most overwhelming advances have been recently achieved in computational seismology at global or regional scale [Komatitsch et al., 2003], [Komatitsch et al., 2004], by employing visco-elastic constitutive models. Anyway, a large set of numerical techniques like finite differences (FD), finite elements (FE), spectral elements (SE) or integral boundary elements (IBE) spurred by the computational power made available by parallel computers, have embraced the area of three-dimensional wave propagation and very interesting results have been achieved particularly in the field of seismic wave propagation [Moczo et al., 2003], [Bielak et al., 1999], [Bielak et al., 2003], [Stupazzini, 2004], [Sanchez-Sesma et al., 1995]. Nevertheless, nearly all models used until recently in seismology for predicting ground motion induced by earthquakes have been based on the assumption of linear elastic behavior of soil. Only few authors, in recent times, have shown the capabilities of taking into account non linear soil behavior in parallel finite element code for large-scale applications [Xu et al., 2003].

This lack of numerical tools capable to solve solid dynamic problems in non-linear field is basically due to three main factors: (i) for years non-linear soil amplification has been routinely taken into account in geotechnical engineering practice by employing simple non-linear elasticity [Seed et al., 1969], (ii) the lack of satisfying constitutive model able to capture time rate dependence in the non linear field and finally (iii) the scarce evidence of non linear effects in the observed motion, other than liquefied sites.

However in the last decades, the second and the third issues have changed completely their perspectives: the viscoplastic constitutive model has grown significantly in importance among the solid dynamic scientific community, thanks to the effort devoted by many authors [Pastor et al., 1995], [Di Prisco et al., 2002], [Marssouli et al., 2003], and seems to be one of the most promising models in order to study accurately non linear propagation phenomena with numerical scheme based on explicit time advancing discretization. Concerning the third issue, a number of accelerograms have been recorded during strong earthquakes that have made possible to infer non linear response of soil, like reduction of shear wave velocity and the increase of soil damping with increasing load [Hardin et al., 1972]. Reliable records of events available for the study of non linear site effects have been provided in a particular seismic region in Taiwan (SMART, Strong Motion Accelerograph Array in Taiwan) [Abrahimson et al., 1987], [Chang et al., 1989], [Wen, 1994], [Beresney, 1995]. Furthermore, the nonlinear behavior of soil during strong earthquakes, like Hyogoken-Nambu 1995 [Sato et al., 1996] or Northridge 1994 [Bardet et al., 1996], [Trifunac et al., 1996], have been demonstrated by different motion recorded. Finally, in some strong ground motions accelerograms, amplitude deamplification as well as changes in resonance frequency have been identified [Archuleta, 1998], [Archuleta et al., 2000].

In this study a parallel spectral element methodology for modeling 2D and 3D wave propagation problems is equipped with a viscoplastic constitutive model in order to grant the flexibility of a finite element technique in dealing with non linear constitutive model, together with the accuracy of spectral methods for propagating waves in heterogeneous
earth media. Due to the innovative aspects of a SE technique for dynamic analyses (for the static case see, for instance, Cividini [Cividini et al., 1997]) dealing with non linear constitutive model, a quite large dataset of comparisons with a very simple 2D study case already developed in literature with FEM based code [Marsouet al., 2003] has been performed and illustrated in the following.

2. GeoELSE

The computational code employed to perform the numerical analyses is named GeoELSE (Geo ELasticity by Spectral Elements [Maggio et al., 2001], [Stupazzini, 2004], [Maggio et al., 2005]). This is a Spectral Element [Faccioli et al., 1997], [Komatitsch, 1998] code for the study of wave propagation phenomena in 2D or 3D complex domains. The recently presented version of the code includes: (i) the capability of dealing with fully unstructured computational domains and (ii) the parallel architecture. While the former feature allows to treat problems involving complex geometries, the second is the natural approach for large scale applications. Before describing the numerical analyses performed and geotechnical problem taken into consideration, it seems worth to briefly introduce the key features of the spectral element method adopted and then to outline the constitutive viscoplastic model introduced.

3. Formulation of the problem

We consider an elastic medium occupying the finite region \( \Omega \subset \mathbb{R}^d \) (\( d = 2 \) or 3 number of space dimension) with boundary \( \Gamma = \Gamma_N \cup \Gamma_D \), where \( \Gamma_D \) is the portion where the displacement vector \( u \) is controlled and \( \Gamma_N \) the portion subject to external forces (tractions). Through the principle of virtual work, the dynamic equilibrium problem for the medium can be stated in the following weak, or variational form [Zienkiewicz et al., 1989]:

\[
\begin{align*}
\frac{\partial^2}{\partial t^2} \int_{\Omega} \rho \cdot v \cdot u \, d\Omega + \int_{\Omega} \sigma_{ij}(u) \varepsilon_{ij}(v) \, d\Omega &= \int_{\Gamma_N} f \cdot v \, d\Gamma + \int_{\Gamma_D} t \cdot v \, d\Gamma, \\
\end{align*}
\]

(1)

where \( t \) is the time, \( \rho = \rho(x) \) the material density, \( \sigma_{ij} \) the stress tensor, \( \varepsilon_{ij} \) the small-strain tensor, \( f=f(x,t) \) a known body force distribution, \( t=t(x,t) \) the vector of external traction prescribed on \( \Gamma_D \), and \( v=v(x) \) is a generic function (candidate to represent admissible displacements) chosen so that all integrals in (1) are finite. Summation on repeated indices will be understood, unless otherwise specified.

The previous equation must be supplemented by suitable conditions prescribing the value of both \( u \) and \( \partial u/\partial t \) for all \( x \in \Omega \) at the initial time \( t=0 \). Concerning the boundary conditions, for simplicity, but without loss of generality, displacements have been assumed to vanish on \( \Gamma_D \).

The stress and strain tensors in (1) are related to the displacement by Hooke’s law

\[
\sigma_{ij}(u) = \lambda \varepsilon_{ij}(u) + 2\mu \varepsilon_{ij}(u)
\]

(2)

and

\[
\varepsilon_{ij}(u) = \frac{1}{2} (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})
\]

(3)

where \( \lambda \) and \( \mu \) are the Lamé elastic coefficient and \( \delta_{ij} \) is the Kronecker delta, such that \( \delta_{ij} = 1 \), if \( i=j \), and \( \delta_{ij} = 0 \), otherwise.

In addition to the “classical” solution of the standard differential formulation of the elastodynamic problem, the variational approach can provide other significant solutions, like those with discontinuous gradient arising in heterogeneous media where \( \rho, \lambda, \mu \) are bounded functions, not necessarily continuous (for instance, they can be piecewise constant). The Galerkin method stemming from (1) is the basis for the spectral element method advocated in finite element methods, as well as in this paper.

3.1. Space discretization with spectral elements

An appropriate numerical solution of (1) can be achieved with discretization in space and time domain. Herein, the former is done via spectral elements and the latter via finite differences.

The spectral element method (SEM) is usually regarded as a generalization of the finite element method (FEM) based on the use of high order piecewise polynomial functions. The crucial aspect of the method is the capability to provide an arbitrary increase of accuracy simply enhancing the algebraic degree of these functions (the spectral degree). On practical ground, this operation is completely transparent for the users, who limit themselves to choose the spectral degree at runtime, leaving to the computational code the task to build up suitable quadrature points and new degrees of freedom. Obviously, increasing the spectral degree has also the effect to raise the computational effort of the problem.

On the other hand, one can also play on the grid refinement to improve the accuracy of the numerical solution, thus following the standard finite element approach. Spectral elements are therefore a so-called “h-p” method [Faccioli et al., 1996],
where “h” refers to the grid size and “p” to the degree of polynomials.

In detail, the SEM adopted in GeoELSE is essentially based on the formulation proposed by Fac-ccioli [Facccioli et al., 1997]. The approximate numerical solution, \( u_N \), as well as the admissible displacement \( v_N \), are chosen as piecewise polynomial functions of degree \( N \), continuous in the whole domain. To simplify the notation, the set of such polynomials is denoted by \( V_N \), and the subset of those vanishing on \( \Gamma_D \) by \( V_N^0 \). Then Eq. (1) is approximated by the following “semi-discrete” formulation:

\[
\forall t \in (0,T), \ find \ u_N = u_N(x,t) \in [V_N^0]^d, \quad d = 1 \ldots 3
\]

such that:

\[
\frac{\partial^2}{\partial t^2} \int_{\Omega} \left[ p u_N + v_N \right] d\Omega + \frac{\partial}{\partial t} \int_{\Gamma} ^{\{x_N\}} v_N d\Gamma + \int_{\Omega} ^{\{x_N\}} f v_N d\Omega \quad \text{for all } v_N \in [V_N^0]^d.
\]

The previous piecewise polynomial function can be defined more precisely thanks to the following assumptions: the computational domain \( \Omega \) is decomposed into a family of \( S_H \) non overlapping quadrilaterals \( Q \) (hexahedra in 3D), with typical linear size \( H \) and such that the intersection between two elements is either empty, or a vertex, or an edge, and \( \Omega = \cup S_H \). A fully non-structured domain decomposition was chosen in order to deal with complex computational domains.

Each element \( \Omega_k \) is mapped into a reference element \( \Omega_{ref} \) (the square \([-1,+1]^2\), in 2D, and the cube \([-1,+1]^3\), in 3D) which is used to evaluate derivatives and integrals in (4).

1. The restriction of \( u_N \) and \( v_N \) to each quadrilateral in 2D (or hexahedra in 3D) is the mapping of polynomials of degree \( N \) with respect to each space variable defined in \( \Omega_{ref} \). \( u_N \) and \( v_N \) are continuous across the element interface.

2. Each derivative function in (4) is evaluated through a linear combination of function values at the so-called Legendre-Gauss-Lobatto (LGL) nodes: the linear combination is expressed by means of the spectral differentiation matrix. Details on the LGL nodes and the spectral differentiation matrix are provided in the sequel.

3. Each volume integral in (4) is evaluated numerically by a high accuracy approximation, specifically the Legendre-Gauss-Lobatto quadrature formula:

\[
\int_{\Omega} f d\Omega = \sum_i f(x_i^{(k)}) \omega_i^{(2)}
\]

where \( x_i^{(k)} \) is the vector of the coordinates of the LGL nodes in \( \Omega_k \) and \( \omega_i^{(2)} \) are the corresponding weights [Canuto et al., 1988], [Davis et al., 1984].

Boundary integrals are evaluated in similar manner.

In order to illustrate the algorithmic aspects of the method, it is important to clarify well the description of LGL points and Lagrange polynomials: the former are the points where the Eq. (4) is actually computed and where numerical integration and differentiation are performed, while the latter are the interpolating polynomials which approximate, the solution \( u_N \) through LGL points.

The distribution of LGL nodes in 2D, for different values of \( N \), is shown in Figure 1.

The LGL nodes \( \{\xi_j\} \) correspond, in a normalized 1D situation, to the zeroes of \( L_N \), the first space derivatives of the Legendre polynomial \( L_N \) of degree \( N \) [Abramovitz et al., 1966] along with the extremes of the interval:

\[
\{\xi_j\} = \{\text{zeros of } L_N\} \cup \{-1,+1\}.
\]

For a degree \( N \) of the Legendre polynomial, one has \( N-1 \) zeroes of \( L_N \), plus the two extremes, i.e. a total of \((N+1)\) LGL nodes in each space direction. For 2D and 3D problems LGL points are \((N+1)^d\) \((d=2,3)\).

Each element \( \Omega_k \) is obtained by a mapping of the master element \( \Omega_{ref} \) through a suitable transformation \( x = F_k(\xi) \), where \( \{x\} \) and \( \{\xi\} \) are the cartesian coordinates in \( \Omega_k \) and \( \Omega_{ref} \) respectively (Figure 2).

Thanks to transformation \( F_k(\xi) \), all computations are performed on the master element \( \Omega_{ref} \) and then brought back into each subdomain \( \Omega_k \) this is the reason why LGL points are shown only in \( \Omega_{ref} \).

\( F_k(\xi) \) is characterized by the Jacobian \( J_k(\xi) \). Acceptable elements are generated when the sign of the Jacobian does not change.

Once the LGL points have been defined, to implement the method we need a set of independent polynomials \( \{p_i\} \subset V_N^0 \); this enables to express the \( i \)th component of the numerical solution as \( u_N^i(t)=p_i(t)p_i^j \), where the \( u_N^i(t) \)'s are the unknowns of the problem. Consequently, the Eq. (4) needs to
be satisfied only by such \( \{ p_j \} \), i.e. taking \( v_N = p_j e_i \), for all \( j \), where \( e_i \) is the \( i \)-th base unit vector.

It turns out that the right choice for the space components of \( \{ v \} \) is the Lagrange polynomials of degree \( N \). In fact, this choice allows on one hand to preserve the exponential accuracy of the spectral method and, on the other hand, to minimize the computational effort. Note that with the choice of Lagrange polynomials, the unknown coefficients \( u_j^0(t) \) coincide with the values of the \( i \)-th component of \( u_N(t) \) at the \( j \)-th grid point.

In 1D, the \( i \)-th Lagrange polynomial can be expressed by the classical product formula as:

\[
\psi_i(\xi) = \frac{(\xi-\xi_1)\cdots(\xi-\xi_{i-1})(\xi-\xi_{i+1})\cdots(\xi-\xi_N)}{(\xi_i-\xi_1)\cdots(\xi_i-\xi_{i-1})(\xi_i-\xi_{i+1})\cdots(\xi_i-\xi_N)}
\]

(7)

taking value 1 in \( \xi_i \) and 0 in \( \xi_j \), for \( i \neq j \), i.e.:

\[
\psi_i(\xi_j) = 0 \quad \text{for} \quad i \neq j \\
\psi_i(\xi_i) = 1 \quad \text{or} \quad \psi_j(\xi_i) = \delta_{ij}
\]

(8)

For example, for \( N = 6 \) one has 7 LGL points and 7 Lagrange polynomials: some of them, in the 1D case, are shown in Figure 3. In this figure the normalized basis \([0,1]\) was used instead of \([-1,1]\).

In 2D or 3D, the interpolation functions are obtained by tensor product of 1D functions of Eq. (7). For example in 2D, the generic spectral element shape function, is:

\[
\Psi_{ij}(\xi,\eta) = \begin{cases} 
1 & \text{for } (\xi=\xi_i) \text{ and } (\eta=\eta_j) \\
0 & \text{for } (\xi \neq \xi_i) \text{ or } (\eta \neq \eta_j)
\end{cases}
\]

(9)

In Figure 4 are depicted three Lagrange polynomials for a simple 2D case.

The spectral domain decomposition formulation can now be written:

\[
\forall t \in (0,T) \quad \text{find} \quad u_N = u_N(\mathbf{x},t) \in [V_N^0]^d, \quad d = 1 \ldots 3
\]

such that:

\[
\sum_{k=1}^K \int_{\Gamma_k} \rho u^{(0)}_i \psi_j^0 d\Gamma + \sum_{k=1}^K \int_{\Gamma_k} \sigma_{im}(u_N)e_im^0 \psi_j^0 d\Gamma = \sum_{k=1}^K \int_{\Gamma_k} f^{(0)} \psi_j^0 d\Gamma + \int_{\Gamma_N} \sigma_{m}(u_N)e_im^0 d\Gamma
\]

(10)

where \( u^{(0)}_i, v^{(0)}_i, t^{(0)}_i \) and \( f^{(0)} \) denote the scalar components of the vectors \( u_N, v_N, t \) and \( f \), respectively; \( \Gamma_N = \Gamma_N \cap \partial \Omega \) denotes the part of the boundary of \( \Omega_N \) shared with \( \Gamma_N \). In (10) the continuity of the displacements across the interface is ensured by assumption. Further, it can be shown that the equations of motion are satisfied by \( u_N \) at all internal nodes, and that tractions at the interfaces can be discontinuous but their “jump” vanishes as \( N \to \infty \).
The traction condition on $\Gamma_N$ is satisfied in a similar manner. A detailed analytical derivation for the 1D cases is given elsewhere [FACCIOLO et al., 1996]. It must be stressed that the actual implementation of the method here illustrated makes direct use of Eq. (10), by picking the Lagrange functions associated with each nodal point.

It is well known that when the “exact” solution $u$ is sufficiently smooth, the numerical solution $u_N$ provided by the proposed method is expected to converge to $u$–$u_N \leq C \exp(-N)$, where $C$ is a constant independent from $H$ and $N$. Therefore, spectral elements converge more rapidly to the exact solution than finite elements, a property known in literature as spectral accuracy.

Assembling the grids of the single elements yields the global LGL grid, that is denoted by $\{x_q\}$ in the sequel. As stated earlier, to the $q$-th node is associated a continuous piecewise polynomial function of degree $N$ that takes value 1 at the node considered and vanishes at the others; this function is denoted by $\Psi_q(x)$. Owing to this choice, the $i$-th displacement component is expressed as:

$$u_N^i(x,t) = u_q^i(t) \Psi_q(x)$$  \hspace{1cm} (11)

where $u_q^i(x,t) = u_q^i(t) \Psi_q(x_q,t)$. Numerical differentiation and integration make also use of LGL nodes. Both are performed in the master element through the transformation $F_k$.

Integration is performed going back to the master element $\Omega_{ref}$ as follows: let the integrals appearing in the reference element be approximated with the quadrature formula:

$$\int f \, d\Omega \cong \sum q f(\xi_q) g(\xi_q) \hat{w}_q$$  \hspace{1cm} (12)

where $\{\xi_q\}$ are the LGL points and the $\hat{w}_q$ are the corresponding weights. From Eq. (10) a system of algebraic equations is obtained by first introducing (11) for the unknown $u_N$. Then, for each scalar component of displacement, as many equations are generated as the number of nodal points. Each equation is obtained from Eq. (10) by choosing $\psi_i(x) = \Psi_i(x)$ ($i=1...d$), where $\Psi_i$ is the Lagrange polynomial associated with $i$-th LGL node. In this way Eq. (10) can be written as an ordinary differential equations system with respect to time:

$$[M] \ddot{U}(t) + [K] U(t) = F(t) + T(t)$$  \hspace{1cm} (13)

where matrices $[M]$ and $[K]$, respectively the mass and the stiffness matrix, arise from the terms on the left-hand side of (10), vectors $F$ and $T$ are due to the contributions of external forces and tractions conditions, respectively, and $U$ are the displacement values $u_N(x_q,t)$ at LGL nodes. The advancement of numerical solution in time will be discussed in the following section.

3.2. Time discretization

The full numerical treatment of the problem at hand requires also the time-discretization of Eq. (13). Several finite difference schemes have been tested for the scalar 2D wave equation [MAGGIO et al., 1994], both implicit and explicit. While the former are unconditionally stable, the latter

![Fig. 4 – Lagrange polynomials for a 2D case (top), associated with different nodal points shown by solid dots, in a simple grid with four elements $\Omega_k$ (bottom). The plots refer to an internal point (left), an interface point (centre), and a cross-point (right), respectively [FACCIOLO et al., 1997].](image-url)
must satisfy the well known Courant-Friedrichs-Levy (CFL) condition:

\[
\Delta t \leq \gamma \frac{\Delta x_{\min}}{c_{\max}}
\]

where \(\Delta x_{\min}\) is the shortest spectral grid spacing, \(c_{\max}\) is an upper bound for the wave propagation velocity, and \(\gamma\) is a positive constant strictly less than 1. Since \(\Delta x_{\min}\) corresponds to the LGL points closest to the boundary, where the spectral grid size is proportional to \(N^{-2}\), the condition for numerical stability can also be written as:

\[
\Delta t \leq \frac{L}{N^2} \frac{\gamma}{c_{\max}}
\]

where \(L\) is the typical linear dimension of \(\Omega_k\). For example if \(N = 5\), \(\gamma \approx 4.3\). The condition (15) shows that the stability requirement on \(\Delta t\) becomes prohibitive for small values of \(L/N^2\) (which corresponds to the \(p\)-version), for which an implicit time discretization is strongly recommended. For the \(h\) and \(h-p\) version, the best tradeoff in terms of accuracy, stability and computational complexity (also in view of parallel computing), is provided by the explicit 2nd order leap-frog scheme (LF2-LF2) [MAGGIO et al., 1994].

\[
\left[\begin{array}{c}
\frac{\partial \mathbf{u}^n}{\partial t} \\
\frac{\partial \mathbf{u}^n}{\partial t} 
\end{array}\right] = 2\left[\begin{array}{c}
\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1} \\
\mathbf{u}^{n+1} - \mathbf{u}^{n-1}
\end{array}\right] + O(\Delta t^2) \\
\left[\begin{array}{c}
\frac{\partial \mathbf{u}^n}{\partial t} \\
\frac{\partial \mathbf{u}^n}{\partial t}
\end{array}\right] = \frac{1}{2\Delta t} \left(\mathbf{u}^{n+1} - \mathbf{u}^{n-1}\right) + O(\Delta t^2)
\]

Finally, it is worth summarizing the crucial aspects of the here adopted SEM technique, compared with a classical FEM: (i) spectral elements converge more rapidly to the exact solution or, equivalently, it is more accurate for a given number of nodal points (spectral accuracy), when the exact solution is sufficiently smooth; (ii) the polynomial order of the method can be set at run-time, without modifying the computational grid like in FEM; (iii) the mass matrix is diagonal for construction. Thanks to this latter fact the inversion needed into an explicit time discretisation come up to be trivial in terms of computational effort.

4. GeoELSEvp

The kernel described up to now follows the most simple way of interpreting the dynamic response of soils, that consists in defining an elasto-viscous model in which the elastic stresses are added to the viscous ones and both contributions are obtained by assuming the linearity of the constitutive relationship. By following such an approach, the coupling between the shear and the volumetric components is missing, and, consequently, these models are not capable of capturing all the phenomena associated to the coupling between water and soil skeleton, in which soil liquefaction is included. Solely in the last decade non-linear numerical analyses of dynamic problems have been performed. In this field the time factor plays a fundamental role, for this reason an elastoplastic approach is not suitable because the dependency of the mechanical response on the load frequency, during cyclic tests, or on the load period when impulsive tests are analysed, is completely provided by the inertial term. Conversely, the elasto-viscoplastic constitutive models take actually into consideration the dependency of the mechanical behaviour of soils on the time factor, but such a dependency is very well known as far as cohesive soils are concerned but less when soft rocks and granular soils [DI PRISCO et al., 1996] are considered.

A standard Perzyna [PERZYNA, 1963] type elasto-viscoplastic constitutive model for the ideal cohesive-frictional material has been implemented in this spectral element numerical algorithm.

The material mechanical response is assumed to be characterized by the superimposition of an instantaneous elastic strain increment and a delayed plastic strain:

\[
\dot{\mathbf{e}}_y = \dot{\mathbf{e}}_y^e + \dot{\mathbf{e}}_y^p
\]

where \(\dot{\mathbf{e}}_y^e\) is the elastic strain rate tensor and \(\dot{\mathbf{e}}_y^p\) is the viscoplastic one.

The stress rate is given by

\[
\frac{\partial \sigma_j}{\partial t} = D_{jph} \left( \frac{\partial \epsilon_{jk}}{\partial t} - \frac{\partial \epsilon_{ph}^p}{\partial t} \right)
\]

where \(D_{jph}\) is the elastic constitutive tensor (Eqs. 2 and 3) and \(\epsilon_{jk}\) the strain tensor. The viscoplastic strain increment \(\dot{\epsilon}_{ij}^p\) [PERZYNA, 1963] may be defined in the following way:

\[
\frac{\partial \epsilon_{ij}^p}{\partial t} = \gamma \left( \phi(f) \right) \frac{\partial g}{\partial \sigma_j}
\]

where \(f\) is the yield function, \(g\) the plastic potential, \(\phi(f)\) the viscous nucleus and \(\gamma\) is a constitutive parameter, that influences the strain rate and consequently the rapidity with which the asymptotic strain value is reached. In above, \((\cdot)\) are the Macaulay brackets,

\[
(\cdot) = \begin{cases} \cdot & \text{if } \cdot \geq 0, \\ 0 & \text{otherwise}. \end{cases}
\]

The plastic potential \(g\) defines the direction of the viscoplastic strain increment tensor and the yield function \(f\) influences its modulus by means of the viscous nucleus \(\phi(f)\).
By using the present flow rule, it is not necessary to introduce the consistency rule, as the viscoplastic strain rate modulus depends directly on the function describing the viscous nucleus and not on the plastic multiplier. The yield function may be positive or negative, without any constraint, i.e. the stress state may be external or internal to the yield locus. It is important to underline that from a numerical point of view, this approach allows to avoid the loading-unloading criterion definition; as a consequence the numerical implementation of this kind of constitutive models into FEM and SEM codes is more simple.

Among the several alternative choices for the function $\phi(f)$, it has been used the following

$$\phi(f) = \left( \frac{f - f_0}{f_0} \right)^N$$

(20)

where $N$ is a model parameter, and $f$ is the yield function. The value $f_0$ characterizes the stress level below which no flow occurs. In the case of a Von Mises material, $f$ is taken as the effective stress $\bar{\sigma}$

$$f = \bar{\sigma} = \sqrt{\frac{3}{2}} s_{ij} s_{ij}$$

(20a)

where $s_{ij}$ is the deviatoric stress tensor. The size of the yield surface varies according to the following suitable hardening/softening law:

$$\frac{\partial \bar{\sigma}_0}{\partial t} = H \frac{\partial \bar{\varepsilon}^p}{\partial t}$$

(21)

where $H$ is the hardening modulus.

For the sake of simplicity, in the following an associated flow rule has been assumed and the accumulated viscoplastic strain is defined as follows:

$$d\bar{\varepsilon}^p = \sqrt{\frac{3}{2}} \frac{d\bar{\varepsilon}^p}{(d\varepsilon^p_{ij})^2)}}^{1/2}$$

(22)

where $d\varepsilon^p_{ij} = d\varepsilon^p_{ij} - (d\varepsilon^p_{ii}/3) \delta_{ij}/3$.

5. Numerical examples

Two different “ideal tests” will be discussed and numerically analysed. Both the first and the second are biaxial tests in which the specimen is subjected to an imposed upper bound displacement characterized, respectively, by a (i) rapid compression wave and (ii) a tensile shock wave. All study cases are characterized by an ideal material with the following properties: Young’s modulus $E = 8 \times 10^7$ Pa, Poisson’s ratio $\nu = 0.3$, an initial yield stress $\bar{\sigma} = 5 \times 10^5$ Pa and a softening modulus of $H = -E/10$. Parameters of Perzyna’s model are $\gamma = 20$ s$^{-1}$ and $N = 1$. Density is $\rho = 2000$ kg/m$^3$.

5.1. Biaxial compression test

The aim of this section is the analysis of a bidimensional specimen subjected to a rapid wave loading on its upper face. The study case is sketched in Figure 5 and it consists of a square 1 m wide, but for the symmetry only one half has been considered in the numerical analyses. The imposed boundary conditions are the following:

(i) on $\Gamma_1$ $u_x = u_y = 0$

(ii) on $\Gamma_2$ the symmetry results on $u_x = 0$

(iii) $\Gamma_3$ is a stress free boundary ($\sigma_{xx} = 0$ and $\sigma_{xy} = 0$)

(iv) finally, displacement at $\Gamma_4$ is given by (Figure 6a):

$$u_x = 0$$

$$u_y = u_d(t),$$

where $u_d(t)$ is given by:

$$u_d(t) = 0.06667 \cdot t$$

$$= 0.01 \quad t > t_f,$$

where $t_f$ has been taken as 0.15 s.

The numerical simulations (Fig. 7) show clearly that the strain localization is captured by GeoEL-SEvp both playing on the grid size (“h”) or on the spectral degree (“p”). This feature must be considered like a fundamental one into a SE technique and it is preserved even if a non-linear constitutive model is introduced. The analysis has been performed with two different structured meshes: the “fine” mesh is characterized by a large number of elements (200 el.) and a spectral degree (SD) equal to 2, that can be considered as a low interpolation degree for a SE technique. The second mesh, “coarse”,

[Diagram of a biaxial compression test]
discretizes the domain with only 8 elements but the spectral degree is chosen equal to 10. It is evident from Figure 7c and Figure 7d that almost the same results in terms of accumulated viscoplastic strain are obtained in these two different analyses (snapshot taken at t = 0.15 s. of Fig. 6).

Thanks to the capabilities of GeoELSE kernel to handle unstructured domain it is possible to study the same problem with the two very different unstructured meshes depicted in Figure 8a and b. Differences between the two meshes lay on (i) the number of elements adopted, (ii) the spectral degree and (iii) the “orientation” of the general pattern.

It can be stated that even with an unstructured mesh (Fig. 8) GeoELSEvp is able to reproduce the same viscoplastic strain pattern obtained with the...
structured meshes (Figure 7). Obviously, due to the mesh dependence phenomena, the stencil of the spectral mesh triggers the shear band in different ways (see Figs. 8c and d).

This kind of problem is observed when strain localisation is numerically analysed. To avoid mesh dependence of solution, viscoplastic constitutive models can be employed. Unfortunately, as already shown by Sluys [SLUYS, 1992] and by di Prisco [di PRISCO et al., 2003], this approach is not always sufficient to avoid mesh dependence when the strain softening regime is considered. One possible solution will be the implementation of nonlocal constitutive model into GeoELSEvp kernel as it has already been successfully introduced in finite difference and finite element codes [di PRISCO et al., 2002], [di PRISCO et al., 2003]. It is important to underline that the viscoplastic constitutive model is not capable of avoiding numerical solution mesh dependency due to the particular viscous nucleus definition as well as the viscous constitutive parameter value. In fact, the constitutive parameter $\gamma$ represents the characteristic time which defines the evolution of the material microstructure on the material mechanical response, this is experimentally calibrated by means of creep tests and not on the localization profile and, as a consequence, not to regularize the numerical solution.

Figure 10 shows the load-displacement curves obtained by means of the four meshes previously considered (Figs. 7 and 8). Up to 0.007 m of di-

![Figure 10](image_url)

**Fig. 10** – Load-displacement curves obtained during a strain softening regime; data are filtered with a low-pass 300 Hz filter.

**Fig. 11** – Load-displacement curves obtained during a strain hardening regime; data are filtered with a low-pass 300 Hz filter.
placement, all results seem to coincide; subsequently the curves suddenly differ because of the mesh dependence.

On the other hand, the results obtained with the same analysis and hardening behavior are depicted in Figure 11; it is evident that all the curves coincide, and therefore, mesh dependence disappears in this case. Thanks to the assumption of associated flow rule, the strain pattern obtained in hardening regime does not show a clear localization direction (Figure 9). It is worth noting that the snapshot scale adopted in softening or hardening regime does not coincide.

5.2. Strain localization in 2D specimen

The second example is taken from literature and has been already analysed with a FEM code, based on a velocity-stress formulation, by Mabssout and Pastor [MABSSOUT et al., 2003].

The bidimensional specimen of Figure 5 is subjected to a shock wave on its upper boundary. The imposed displacement at $\Gamma_4$ is given by (Figure 6b):

$$
u_x = 0$$
$$u_y = -u_0(t),$$

where $u_0(t)$ is given by:

$$u_0(t) = \begin{cases} 
1 \cdot t & 0 \leq t \leq t_f, \\
5 \times 10^{-3} & t > t_f,
\end{cases}$$

where $t_f$ has been taken as $5 \times 10^{-3}$ s.

The shear band starts from point A (Fig. 5) when the wave is reflected at the base. The amplitude of the wave has been chosen such that the stresses do not reach the yield surface until the wave is reflected at the bottom, which happens at $t = 4.5 \times 10^{-3}$ s. Hereafter, the stress...
doubles, the stress path crosses the yield surface, and the strain localizes in the form of a shear band which is incepted at point A (Fig. 5). Inclination of the shear band is 58°, which agrees well with analytical solution.

The numerical results obtained in terms of deformed mesh are shown in Figure 12, in particular: (a) fine mesh with SD equal to 1, (b) 8 elements structured mesh with SD equal to 7, (c) 2 elements mesh with SD equal to 13 and (d) 34 elements unstructured mesh with SD equal to 3. While the corresponding contours of the accumulated viscoplastic strain are given respectively in Figure 12 (e), (f), (g) and (h). Though the meshes considered are extremely different, among each others, both from the grid size point of view and from the spectral degree adopted, the localization of shear band shows a very similar shape. Even if this second example seems to highlight no mesh dependence, it is worth to stress out the fact that the snapshot are captured at 9 ms. Therefore if the simulation went on, shear band will increase and once again mesh dependence could be observed.

Finally, it is crucial to underline, once again, the capability of GeoELSEvp to provide an arbitrary increase of accuracy, achieved just increasing the spectral degree at run time and preserving the user from any remeshing operation. The code is encharged to fulfill the task to build up a suitable quadrature points and new degree of freedom. This aspect is clearly shown in Figure 13, where the results obtained analyzing only the 8 elements mesh with different spectral degree are reported. The spectral degree 4 (Fig. 13a) is already capable of capturing the shear band occurrence, anyway going up to spectral degree 7 (b) and spectral degree 10 (c), the solution is improved. On the contrary, from spectral degree equal to 10 up to spectral degree 13 (Fig. 13d) no further solution refinement is recognizable.
Conclusions

GeoELSE is a Spectral Elements code for the study of wave propagation phenomena in 2D or 3D complex domains. One of the peculiarities of the formulation of SEM adopted in such a numerical code is the strong similarity with the FEM. This allows the authors to introduce a viscoplastic constitutive model quite easily into the GeoELSE kernel and testing its capabilities to handle non-linear behavior with time rate dependence.

The very simple constitutive law introduced for the soil is justified by the goal of the research work which does not consist in reproducing a real case but in illustrating the capability of the approach. The “ideal test” numerically simulated concerns the localization of a soil specimen characterized by a marked strain softening. The 2D examples provided show that the SEM version adopted is able to capture non linear phenomena with good accuracy. In particular, it is worth noting that (i) the shear band formation (both in rapid wave and shock wave) is captured correctly and viscoplastic strains cross the elements, (ii) solution seems to improve playing both on the refinement (“h”) of the mesh or increasing the spectral degree (“p”) without providing any refinement of the computational domain.

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References


The solution of acoustic and elastic wave equation. Geophysics, 61, pp. 1160-1174.


GeoELSEvp: un approccio agli elementi spettrali per problemi dinamici elasto-viscoplastici

Sommaro

In un codice di calcolo per lo studio di problemi 2D e 3D di propagazione di onde, basato su di un metodo ad elementi spettrali, è stato introdotto un legame costitutivo di tipo elasto-viscoplastico. La tecnica numerica messa a punto è tale da garantire sia la flessibilità di un metodo ad elementi finiti per l’analisi in campo non lineare, che l’accuratezza di un metodo spettrale per la propagazione di onde. Il codice proposto è veloce e di semplice utilizzo, grazie alla minima interazione richiesta. L’implementazione della tecnica numerica messa a punto viene illustrata e applicata a due casi bidimensionali.